The weighted means of these, weights being given according to their probable errors, will be new values of the personal errors Δa, Δb, Δc of the judges and applying these we obtain new values for the merits. It is easily seen however that this can only affect the 0.01 and though theoretically speaking an infinite number of approximations is required to obtain the most probable values of the merits we may safely regard the second approximation as sufficient.

ON THE CIRCULAR POINTS AT INFINITY.

By E. D. ROE, JR., Associate Professor of Mathematics in Oberlin College, Oberlin, Ohio.

I. The Coördinate System. In the following discussion, in addition to the usual Cartesion coördinates, homogeneous point and line coördinates will be used. They are related to Cartesian coördinates as follows :*

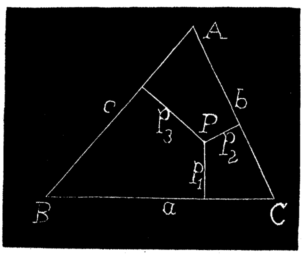

Fig. 1.

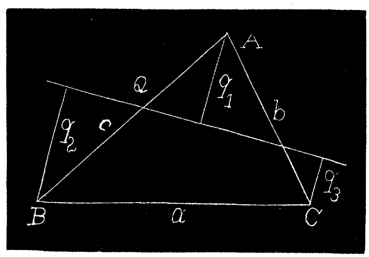

Fig. 1′.

In figure 1, p_1, p_2, p_3 are the perpendicular distances of a point P, from the sides of the coördinate triangle.

In figure 1′, q_1, q_2, q_3 are the perpendicular distances of a line Q, from the vertices of the coördinate triangle.

The three point coördinates of P are expressed as follows :

$$\begin{aligned} \rho x_1 &= p_1 \varkappa_1 \\ \rho x_2 &= p_2 \varkappa_2 \\ \rho x_3 &= p_3 \varkappa_3 \end{aligned} \tag{1}$$

The three line coördinates of Q are expressed as follows :

$$\begin{aligned} \sigma u_1 &= q_1 \lambda_1 \\ \sigma u_2 &= q_2 \lambda_2 \\ \sigma u_3 &= q_3 \lambda_3 \end{aligned} \tag{1$'$}$$

The $\varkappa$'s and λ's are six constants which might be chosen at pleasure, but for convenience are chosen in a particular way.

*For a fuller treatment see Clebsch, Vorlesungen ueber Geometrie, S. 27-29, S. 62-78.

In Cartesian

point coördinates the equations of the sides of the triangle may be

$$\begin{aligned} a_1x+b_1y+c_1&=0 \text{ for } BC\\ a_2x+b_2y+c_2&=0 \text{ for } CA\\ a_3x+b_3y+c_3&=0 \text{ for } AB \end{aligned} \qquad (2)$$

line coördinates the equations of the opposite vertices will be

$$\begin{aligned} A_1u+B_1v+C_1&=0 \text{ for } A\\ A_2u+B_2v+C_2&=0 \text{ for } B\\ A_3u+B_3v+C_3&=0 \text{ for } C \end{aligned} \qquad (2)'$$

We have then

$$p_1 = \frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}, \quad p_2 = \frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}, \quad p_3 = \frac{a_3x+b_3y+c_3}{\sqrt{a_3^2+b_3^2}} \qquad (3)$$

$$q_1 = \frac{A_1u+B_1v+C_1}{C_1\sqrt{u^2+v^2}}, \quad q_2 = \frac{A_2u+B_2v+C_2}{C_2\sqrt{u^2+v^2}}, \quad q_3 = \frac{A_3u+B_3v+C_3}{C_3\sqrt{u^2+v^2}} \qquad (3)'$$

We choose

$$\varkappa_1=\sqrt{a_1^2+b_1^2},\ \varkappa_2=\sqrt{a_2^2+b_2^2},\ \varkappa_3=\sqrt{a_3^2+b_3^2}$$

$$\lambda_1=C_1,\ \lambda_2=C_2,\ \lambda_3=C_3$$

and write

$$\begin{aligned} \rho x_1&=a_1x+b_1y+c_1\\ \rho x_2&=a_2x+b_2y+c_2\\ \rho x_3&=a_3x+b_3y+c_3 \end{aligned} \qquad (A)$$

$$\begin{aligned} \sigma u_1&=A_1u+B_1v+C_1\\ \sigma u_2&=A_2u+B_2v+C_2\\ \sigma u_3&=A_3u+B_3v+C_3 \end{aligned} \qquad (A)'$$

Solving these equations, and writing r for (abc),

$$x = \frac{\rho(A_1x_1+A_2x_2+A_3x_3)}{r}, \quad y = \frac{\rho(B_1x_1+B_2x_2+B_3x_3)}{r}, \quad 1 = \frac{\rho(C_1x_1+C_2x_2+C_3x_3}{r}$$

$$u = \frac{\sigma r(a_1u_1+a_2u_2+a_3u_3}{r^2}, \quad v = \frac{\sigma r(b_1u_1+b_2u_2+b_3u_3}{r^2}, \quad 1 = \frac{\sigma r(c_1u_1+c_2u_2+c_3u_3}{r^2}$$

or

$$x = \frac{A_1x_1+A_2x_2+A_3x_3}{C_1x_1+C_2x_2+C_3x_3}, \quad y = \frac{B_1x_1+B_2x_2+B_3x_3}{C_1x_1+C_2x_2+C_3x_3} \qquad (5)$$

$$u = \frac{a_1u_1+a_2u_2+a_3u_3}{c_1u_1+c_2u_2+c_3u_3}, \quad v = \frac{b_1u_1+b_2u_2+b_3u_3}{c_1u_1+c_2u_2+c_3u_3} \qquad (5)'$$

Upon our choice of the $\varkappa$'s and λ's depends the following important result:

(6)

$$\rho\sigma(u_1x_1+u_2x_2+u_3x_3)=r(ux+vy+1).$$

Hence $u_1x_1+u_2x_2+u_3x_3$ vanisnes whenever $ux+vy+1$ vanishes. We may note that

(7)

$C_1x_1+C_2x_2+C_3x_3=0$ is the equation of the line at infinity. It gives the condition that $x=y=\infty$.

(7)′

$c_1u_1+c_2u_2+c_3u_3=0$, is the equation of the origin of coördinates. It gives the condition that $u=v=\infty$.

II. The Circular Points.*

A. Proof that all circles whatsoever pass through two points at infinity.

The equations of any two circles may be written,

$$\begin{aligned}x^2+y^2+2gx+2fy+c=0.\\ x^2+y^2+2g'x+2f'y+c'=0.\end{aligned}\tag{8}$$

In homogeneous point coördinates,

$$x=\frac{A_1x_1+A_2x_2+A_3x_3}{C_1x_1+C_2x_2+C_3x_3}=\frac{A}{C}$$

by (5) and our equations become,

$$y=\frac{B_1x_1+B_2x_2+B_3x_3}{C_1x_1+C_2x_2+C_3x_3}=\frac{B}{C}.$$

$$\begin{aligned}A^2+B^2+2gAC+2fBC+cC^2=0.\\ A^2+B^2+2g'AC+2f'BC+c'C^2=0.\end{aligned}\tag{9}$$

The lines passing through their points of intersection are, by subtraction,

$$C[2(g-g')A+2(f-f')B+(c-c')C]=0.\tag{10}$$

Of these the line $C=0$, is the one that interests us. It is the equation of the line at infinity. The infinite points are found by solving $C=0$, with the equation of either circle, and thus we find them from $C=0$, and $A^2+B^2=0$, or in Cartesian coördinates from the equation of the line at infinity and $x^2+y^2=0$; this would give the same solution always for any two circles; therefore every circle passes through two points at infinity.

B. Cartesian equation of the points in line coördinates.

The equations of the points in Cartesian line coördinates may be readily obtained.

*See Clebsch, Vorlesungen ueber Geometrie, S. 145-149. Salmon's Conic Sections, pages 238, 325. Fiedler's Salmon, Analytische Geometrie der Kegelschnitte, S. 208.

As $x=\frac{A}{C}$, $y=\frac{B}{C}$, the equation of a point $ux+vy+1=0$, becomes

$$Au+Bv+C=0.$$

$x+yi=0$, gives for any point on the line $\quad A+Bi=0.$
We must then have for one of the points $Au+Bv+C=0$, all true.

$$A+Bi=0$$
$$C=0$$

$$\text{Hence} \begin{vmatrix} u & v & 1 \\ 1 & i & 0 \\ 0 & 0 & 1 \end{vmatrix} =0, \text{ or } ui-v=0,$$

similarly for the other point $-ui-v=0$.

For the pair we have $u^2+v^2=0$. (11)

C. Coördinates of the circular points.

1. *Homogeneous rectangular coördinates.* We saw that we could find the coördinates of the circular points by solving the equations of the line at infinity, $x^2+y^2=0$. The equation of the line at infinity is $0x+0y+c=0$. $x^2+y^2=(x+yi)(x-yi)=0$, a pair of imaginary straight lines through the origin. We will find the intersections of a line $ax+by+c=0$ with $x+yi=0$, and $x-yi=0$.

Solving $ax+by=-c$, we get $x=\frac{-ci}{ai-b}$.

$$x+yi=0 \qquad\qquad y=\frac{c}{ai-b}.$$

Or $\frac{x}{-i}=\frac{y}{1}=\frac{c}{ai-b}$. If now we introduce a third coördinate c to make our rectangular coöordinates homogeneous, and consider the ratios of x, y, and c as coördinates, we see that the coördinates of one imaginary circular point are given by $\quad x:y:c=-i:1:0,$
and of the other by $\quad x:y:c=i:1:0.$ $\qquad$ (12)

2. *Homogeneous point coöordinates.* The coördinates of the circular points however assume a more convenient form when expressed in the general point coödinates. We shall obtain them in proving the following highly interesting proposition:

A circle, with fixed center in the finite region, whose radius becomes indefinitely great, degenerates into the two circular points at infinity.

We will obtain the equation of the circle in homogeneous line coördinates. We express that a line u is always at a distance r from the point x', the center of the circle.

Let $u_1x_1+u_2x_2+u_3x_3=0$ be the point equation of the line [see (6)].

This by (4) is

$$u_1(a_1x+b_1y+c_1)+u_2(a_2x+b_2y+c_2)+u_3(a_3x+b_3y+c_3)=0$$

or $(u_1a_1+u_2a_2+u_3a_3)x+(u_1b_1+u_2b_2+u_3b_3)y+(u_1c_1+u_2c_2+u_3c_3)=0.$ (13)

Putting this in cosine form, and taking the square of the distance from $x'y'$ to the line equal to r^2, we get,

$$\frac{[(u_1a_1+u_2a_2+u_3a_3)x'+(u_1b_1+u_2b_2+u_3b_3)y'+(u_1c_1+u_2c_2+u_3c_3)]}{(u_1a_1+u_2a_2+u_3a_3)^2+(u_1b_1+u_2b_2+u_3b_3)^2}=r^2.$$

or (14)*

$$\frac{(u_1x_1'+u_2x_2'+u_3x_3')^2}{\varkappa_1^2u_1^2+\varkappa_2^2u_2^2+\varkappa_3^2u_3^2-2\varkappa_1\varkappa_2u_1u_2\cos C-2\varkappa_2\varkappa_3u_2u_3\cos A-2\varkappa_3\varkappa_1u_3u_1\cos B}=r^2$$

The reductions in the denominator depend on the following :

$$a_1^2+b_1^2=\varkappa_1^2,\text{ etc.}\quad a_1a_2+b_1b_2=\varkappa_1\varkappa_2\left(\frac{a_1b_2}{\varkappa_1\varkappa_2}+\frac{b_1b_2}{\varkappa_1\varkappa_2}\right)$$

$$=\varkappa_1\varkappa_2(\cos\alpha_1\cos\alpha_2-\sin\alpha_1\sin\alpha_2)=\varkappa_1\varkappa_2\cos C,\text{ etc.},$$

where α_1, α_2, α_3 are the angles that p_1, p_2, p_3 make with the axis of x, and the origin is taken within the coördinate triangle. (14) is the line equation of the circle. We notice now that the expression in the denominator may be factored, for considering the variables as, $\varkappa_1u_1$, $\varkappa_2u_2$, $\varkappa_3u_3$, its discriminant is

$$\begin{vmatrix} 1 & -\cos C & -\cos B \\ -\cos C & 1 & -\cos A \\ -\cos B & -\cos A & 1 \end{vmatrix}$$

and that this is zero, may be shown as follows :

From trigonometry we have the three equations :

$$\begin{aligned} a-b\cos C-c\cos B&=0\\ -a\cos C+b-c\cos A&=0\\ -a\cos B-b\cos A+C&=0 \end{aligned}$$

whence it follows that the determinant of the coefficients of a, b, and c, vanishes. Put $\varkappa_1u_1=l$, etc. Then

$$l^2+m^2+n^2-2lm\cos C-2mn\cos A-2nl\cos B=(l\alpha+m\beta+n\gamma)(l\alpha'+m\beta'+n\gamma')$$

$$=l^2\alpha\alpha'+m^2\beta\beta'+n^2\gamma\gamma'+lm(\alpha\beta'+\alpha'\beta)+mn(\beta\gamma'+\beta'\gamma)+nl(\alpha\gamma'+\alpha'\gamma),$$

and we must have :

*Compare Salmon's Conic Sections, page 128, Ex. 6.

$$\begin{array}{lll} \alpha\alpha'=1. & \text{Take } \alpha=\cos B-i\sin B, & \gamma=-1, \\ \beta\beta'=1. & \alpha'=\cos B+i\sin B, & \gamma'=-1, \\ \gamma\gamma'=1. & \beta=\cos A+i\sin A, & \\ & \beta'=\cos A-i\sin A, & \end{array}$$

and the first three conditions are satisfied. Also,

$$\alpha\beta'+\alpha'\beta=[\cos(A+B)-i\sin(A+B)+\cos(A+B)+i\sin(A+B)]$$
$$=2\cos(A+B)=-2\cos C.$$
$$\beta\gamma'+\beta'\gamma=-\cos A-i\sin A-\cos A+i\sin A=-2\cos A.$$
$$\alpha\gamma'+\alpha'\gamma=-\cos B+i\sin B-\cos B-i\sin B=-2\cos B.$$

Our expression therefore has the two factors, viz :

$$[(\cos B-i\sin B)\varkappa_1 u_1+(\cos A+i\sin A)\varkappa_2 u_2-\varkappa_3 u_3]=L.$$

$$[(\cos B+i\sin B)\varkappa_1 u_1+(\cos A-i\sin A)\varkappa_2 u_2-\varkappa_3 u_3]=M.$$

Also write $u_1x_1+u_2x_2+u_3x_3=u_x$, then our line equation of the circle may be written

$$LM=\left(\frac{u_{x'}}{r}\right)^2. \qquad (15)$$

If γ becomes indefinitely great $u_{x'}$ does not become indefinitely great, for the x''s are finite, the coördinates of the fixed center, and the u's by (4)$'$ are always finite, since u and v are always finite, and for a line which is moved off to infinity approach zero together. It follows that $\lim_{r=\infty}\left(\frac{u_{x'}}{r}\right)=0$.

Hence the equation of a circle whose radius is infinite, and whose center is in the finite region is in line coördinates,

$$LM=0. \qquad (16)$$

But this is also the equation of a point-pair, and since we have proved that every circle whatsoever contains the two imaginary circular points at infinity, it follows that the two points into which this circle has degenerated are themselves the two imaginary circular points at infinity. As we might just as well have factored our expression LM in two other ways, in which the two angles B and C, or C and A, play the same part as A and B, we may write the coördinates of the two points in the three ways, as follows:

$$\begin{array}{llll} (1). & -\varkappa_1, & \varkappa_2 e^{-iC}, & \varkappa_3 e^{iB}. \\ & -\varkappa_1, & \varkappa_2 e^{iC}, & \varkappa_3 e^{-iB}. \\ (2). & \varkappa_1 e^{iC}, & -\varkappa_2, & \varkappa_3 e^{-iA}. \\ & \varkappa_1 e^{-iC}, & -\varkappa_2, & \varkappa_3 e^{iA}. \\ (3). & \varkappa_1 e^{-iB}, & \varkappa_2 e^{iA}, & -\varkappa_3. \\ & \varkappa_1 e^{iB}, & \varkappa_2 e^{-iA}, & -\varkappa_3. \end{array} \qquad (17).$$

Before going on to the next division of our discussion, we will recall* that if

$$a_{11}x_1^2+a_{22}x_2^2+a_{33}x_3^2+2a_{12}x_1x_2+2a_{23}x_2x_3+2a_{31}x_3x_1=0$$

where $a_{ik}=a_{ki}$ is the point equation of a conic, with non-vanishing discriminant,

$$A_{11}u_1^2+A_{22}u_2^2+A_{33}u_3^2+2A_{12}u_1u_2+2A_{23}u_2u_3+2A_{31}u_3u_1=0$$

where $A_{ik}=A_{ki}$ is the line equation of a conic, with non-vanishing discriminant,

then

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & u_1 \\ a_{21} & a_{22} & a_{23} & u_2 \\ a_{31} & a_{32} & a_{33} & u_3 \\ u_1 & u_2 & u_3 & 0 \end{vmatrix}=0. \qquad (18)$$

is the line equation of the same conic.

$$\begin{vmatrix} A_{11} & A_{12} & A_{13} & x_1 \\ A_{21} & A_{22} & A_{23} & x_2 \\ A_{31} & A_{32} & A_{33} & x_3 \\ x_1 & x_2 & x_3 & 0 \end{vmatrix}=0. \qquad (18)'$$

is the point equation of the same conic.

and that always

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & x_1 & x_1' \\ a_{21} & a_{22} & a_{23} & x_2 & x_2' \\ a_{31} & a_{32} & a_{33} & x_3 & x_3' \\ x_1 & x_2 & x_3 & 0 & 0 \\ x_1' & x_2' & x_3' & 0 & 0 \end{vmatrix}=0 \qquad (19)$$

is the equation of the pair of tangents from the point x' to the same conic.

$$\begin{vmatrix} A_{11} & A_{12} & A_{13} & u_1 & u_1' \\ A_{21} & A_{22} & A_{23} & u_2 & u_2' \\ A_{31} & A_{32} & A_{33} & u_3 & u_3' \\ u_1 & u_2 & u_3 & 0 & 0 \\ u_1' & u_2' & u_3' & 0 & 0 \end{vmatrix}=0 \qquad (19)'$$

is the equation of the pair of points where the line u' cuts the same conic.

III. Fundamental Geometrical Relations Defined in Terms of the Circular Points at Infinity.

A. *The equation of a circle in terms of the circular points.*

Our line equation of the circle (14) may be written :

$$(r^2\varkappa_1^2-x_1^2)u_1^2+(r^2\varkappa_2^2-x_2'^2)u_2^2+(r^2\varkappa_3^2-x_3'^2)u_3^2$$
$$-2(r^2\varkappa_1\varkappa_2\cos B+x_1'x_2')u_1u_2-2(r^2\varkappa_2\varkappa_3\cos A+x_2'x_3')u_2u_3$$
$$-2(r^2\varkappa_3\varkappa_1\cos B+x_3'x_1')u_3u_1=0.$$

Therefore by (18)', its point equation is,

$$\begin{vmatrix} \gamma^2\varkappa_1^2-x_1'^2 & -(\gamma^2\varkappa_1\varkappa_2\cos C+x_1'x_2') & -(\gamma^2\varkappa_2\varkappa_3\cos B+x_1'x_3') & x_1 \\ -(r^2\varkappa_1\varkappa_2\cos C+x_1'x_2') & r^2\varkappa_2^2-x_1'^2 & -(r^2\varkappa_2\varkappa_3\cos A+x_2'x_3') & x_2 \\ -(r^2\varkappa_1\varkappa_3\cos B+x_1'x_3') & -(r^2\varkappa_2\varkappa_3\cos A+x_2'x_3') & r^2\varkappa_3^2-x_3' & x_3 \\ x_1 & x_2 & x_3 & 0 \end{vmatrix}=0.$$

The coefficients of x_1^2 and x_1x_2 changed in sign will be :

*Clebsch, Vorlesungen ueber Geometrie, S. 113.

Of x_1^2, $(r^2\varkappa_2^2-x_2'^2)(r^2\varkappa_3^2-x_3'^2)-(r^2\varkappa_2\varkappa_3\cos A+x_2'x_3')^2$

$$=r^4\varkappa_2^2\varkappa_3^2-r^2(\varkappa_3^2x_2'^2+\varkappa_2^2x_3'^2)-r^4\varkappa_2^2\varkappa_3^2\cos^2A-2r^2\varkappa_2\varkappa_3x_2'x_3'\cos A,$$

of x_1x_2, $2(r^2\varkappa_3^2-x_3'^2)(r^2\varkappa_1\varkappa_2\cos C+x_1'x_2')$

$$+2(r^2\varkappa_2\varkappa_3\cos A+x_2'x_3')(r^2\varkappa_1\varkappa_3\cos B+x_1'x_3')$$

$$=2(r^4\varkappa_1\varkappa_2x_3^2\cos C-r^2\varkappa_1\varkappa_2\cos Cx_3'^2+r^2\varkappa_3^2x_1'x_2'+r^4\varkappa_1\varkappa_2\varkappa_3\cos A\cos B$$
$$+r^2\varkappa_2\varkappa_3x_1'x_3'\cos A+r^2\varkappa_1\varkappa_3x_2'x_3'\cos B).$$

Similarly for the other terms. Put terms containing r^2 on right hand side of equation, divide by r^2, arrange, and reduce, and we get finally,

$$r^2(\varkappa_2\varkappa_3\sin Ax_1+\varkappa_3\varkappa_1\sin Bx_2+\varkappa_1\varkappa_2\sin Cx_3)^2$$
$$=\varkappa_3^2(x_2'x_1-x_2x_1')^2+\varkappa_1^2(x_3'x_2-x_3x_2')^2+\varkappa_2^2(x_1'x_3-x_1x_3')^2$$
$$-2\varkappa_2\varkappa_3(x_3x_1'-x_3'x_1)(x_1x_2'-x_1'x_2)\cos A-2\varkappa_3\varkappa_1(x_1x_2'-x_1'x_2)(x_2x_3'-x_2'x_3)\cos B$$
$$-2\varkappa_1\varkappa_2(x_2x_3'-x_2'x_3)(x_3x_1'-x_3x_1)\cos C.* \qquad (20)$$

From what we did with LM of (14) it is clear that the right member of (20) can be factored. Put $(x_3'x_2-x_3x_2')=l$, etc. We then have as factors,

$$\begin{array}{ll} l\varkappa_1e^{-iB}+m\varkappa_2e^{iA}-n\varkappa_3 & \text{or if } \rho\xi_1=\varkappa_1e^{-iB},\ \rho\xi_1'=\varkappa_1e^{iB}. \\ l\varkappa_1e^{iB}+m\varkappa_2e^{-iA}-n\varkappa_3 & \rho\xi_2=\varkappa_2e^{iA},\ \rho\xi_2'=\varkappa_2e^{-iA}. \\ & \rho\xi_3=-\varkappa_3,\ \rho\xi_3'=-\varkappa_3. \end{array}$$

The factors become,

$$\rho(l\xi_1+m\xi_2+n\xi_3)$$
$$\rho(l\xi_1'+m\xi_2'+n\xi_3')$$

and by supplying the values of l, m, and n, these become

$$\rho\begin{vmatrix} \xi_1 & x_1 & x_1' \\ \xi_2 & x_2 & x_2' \\ \xi_3 & x_3 & x_3' \end{vmatrix}, \qquad \rho\begin{vmatrix} \xi_1' & x_1 & x_1' \\ \xi_2' & x_2 & x_2' \\ \xi_3' & x_3 & x_3' \end{vmatrix}$$

Also the left member is $r^2(C_1x_1+C_2x_2+C_3x_3)^2$ since

$$\varkappa_2\varkappa_3\sin A=\varkappa_2\varkappa_3\sin(\alpha_3-\alpha_2)=\varkappa_2\varkappa_3\left(\frac{a_2b_3-a_3b_2}{\varkappa_2\varkappa_3}\right)=C_1,\ \text{etc.},$$

*Compare Salmon's Conic Sections, page 128, Ex. 6.

and (20) takes the form

$$r^2(C_1x_1+C_2x_2+C_3x_3)^2=\rho^2(\xi xx')(\xi' xx'). \qquad (21)$$

As a check on our determination of the coördinates of the circular points at infinity, let us see if the coördinates, $\rho\xi$, $\rho\xi'$ satisfy this equation of the circle. They certainly reduce the right member to zero, for $(\xi\xi x')=0$, and $(\xi'\xi' x')=0$. They also reduce the left member to zero, for it is reduced to zero, by the coördinates of any infinitely distant points [see (7)]. We thus confirm the results of (17), and (21) is the equation of a circle, whose center is at x', in terms of the coördinates of the circular points at infinity.*

B. General distance formula in terms of the circular coördinates.

The preceding result (21) may be used as a formula for the distance between two points x and x'. It must first however be made homogeneous in all the coördinates. It is clear that,

$$C_1x_1+C_2x_2+C_3x_3=\varkappa\rho^2\begin{vmatrix} x_1 & \xi_1 & \xi_1' \\ x_2 & \xi_2 & \xi_2' \\ x_3 & \xi_3 & \xi_3' \end{vmatrix},$$

for the vanishing of both expressions signifies a line through two infinitely distant points. To determine $\varkappa$, let $x_2=x_3=0$.

Then $C_1x_1=\varkappa\rho^2\begin{vmatrix} x_1 & \xi_1 & \xi_1' \\ 0 & \xi_2 & \xi_2' \\ 0 & \xi_3 & \xi_3' \end{vmatrix}$

$$=\varkappa\rho^2x_1\frac{(-\varkappa_2\varkappa_3(\cos A+i\sin A)+\varkappa_2\varkappa_3(\cos A-i\sin A)}{\rho^2}$$

$$=-2\varkappa x_1\varkappa_2\varkappa_3 i\sin A$$

$$=-2\varkappa x_1 iC_1.$$

$\therefore\ \varkappa=-\dfrac{1}{2i}=\frac{1}{2}i$, and we have

$$C_1x_1+C_2x_2+C_3x_3=\tfrac{1}{2}(i\rho^2)(x\xi\xi')=$$

a constant by the first solution of (4).

By this fact we can make r^2 homogeneous. For

$$r^2=c\frac{(\xi xx')(\xi' xx')}{(x\xi\xi')^2(x'\xi\xi)^2}$$

where c is some constant. To determine it let the distance between a and a' be

*A question might arise as to the constant ρ^2. That can be disposed of as is done in the next division.

unity. Substituting the value of c so obtained we have

$$r^2 = \frac{(\xi x x')(\xi' x x')(a\xi\xi')^2(a'\xi\xi')^2}{(x\xi\xi')^2(x'\xi\xi')^2(\xi a a')(\xi' a a')}, \quad (22)$$

which is seen to be homogeneous and of degree zero in all the coördinates. It is also clear that it is an absolute invariant expression in ternary forms, for on account of the multiplication law of determinants any determinant of the form $(x'y'z')$ in terms of the new variables becomes equal under linear transformation, to $M(xyz)$ where M is the modulus of the transformation, and the transformation equations are:

$$\begin{aligned} \rho x_1' &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ \rho x_2' &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ \rho x_3' &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \; * \end{aligned}$$

Our distance is thus defined projectively with respect to the circular points.

C. *The equation of the pair of lines from a point x to the two circular points.*

These lines will certainly be given by $(xx'\xi)(xx'\xi')=0$. (23).

By (16) the equation of the circular points in line coördinates is

$$\varkappa_1^2u_1^2 + \varkappa_2u_2^2 + \varkappa_3u_3^2 - 2\varkappa_1\varkappa_2u_1u_2\cos C - 2\varkappa_2\varkappa_3u_2u_3\cos A - 2\varkappa_3\varkappa_1u_3u_1\cos B = 0.$$

Now the equation of the pair of tangents from x', to the points will be by (19),

$$\begin{vmatrix} \varkappa_1^2 & -\varkappa_1\varkappa_2\cos C & -\varkappa_1\varkappa_3\cos B & x_1 & x_1' \\ -\varkappa_1\varkappa_2\cos C & \varkappa_2^2 & -\varkappa_2\varkappa_3\cos A & x_2 & x_2' \\ -\varkappa_1\varkappa_3\cos B & -\varkappa_2\varkappa_3\cos A & \varkappa_3^2 & x_3 & x_3' \\ x_1 & x_2 & x_3 & 0 & 0 \\ x_1' & x_2' & x_3' & 0 & 0 \end{vmatrix} = 0. \quad (24)$$

It follows that this determinant is equal to the left member of (23) multiplied by a constant, since their vanishing represents the same geometrical form. Let $\varkappa$ denote this constant. Put $x_1 = a_4$, $x_2' = b_5$, above. $\varkappa_3^2 = c_3$, $x_1 = d_1$, $x_2' = e_2$, below. On the left hand the term containing $x_1^2x_2'^2$ will be represented by $a_4b_5c_3d_1e_2$. The number of inversions of order $j = 3+3+2 = 8$. On the right hand $\varkappa x_1^2 \begin{vmatrix} x_2' & \\ & \xi_3 \end{vmatrix} \begin{vmatrix} x_2' & \\ & \xi_3' \end{vmatrix} = \dfrac{\varkappa x_1^2 x_2' \varkappa_3^2}{\rho^2}$.

*The result of this division is found in Klein's First Lecture, Winter Semester, 1889-90, on the "Nicht-Euclidsche Geometry," S. 40.

$$\therefore\ x_1^2 x_2'^2 \varkappa_3^2 = \frac{\varkappa}{\rho^2} x_1^2 x_2'^2 \varkappa_3^2. \quad \therefore\ \varkappa = \rho^2,$$

and we obtain the interesting result in determinants,*

$$\begin{vmatrix} \varkappa_1^2 & -\varkappa_1\varkappa_2\cos C & -\varkappa_1\varkappa_3\cos B & x_1 & x_1' \\ -\varkappa_1\varkappa_2\cos C & \varkappa_2^2 & -\varkappa_2\varkappa_3\cos A & x_2 & x_2' \\ -\varkappa_1\varkappa_3\cos B & -\varkappa_2\varkappa_3\cos A & \varkappa_3^2 & x_3 & x_3' \\ x_1 & x_2 & x_3 & 0 & 0 \\ x_1' & x_2' & x_3' & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} x_1 & x_1' & \varkappa_1 e^{-iB} \\ x_2 & x_2' & \varkappa_2 e^{iA} \\ x_3 & x_3' & -\varkappa_3 \end{vmatrix} \cdot \begin{vmatrix} x_1 & x_1' & \varkappa_1 e^{iB} \\ x_2 & x_2' & \varkappa_2 e^{-iB} \\ x_3 & x_3' & -\varkappa_3 \end{vmatrix} \qquad (25)$$

D. The angle between the lines.

Let us take the four lines,

$$x+iy=0. \quad 1. \qquad x-iy=0. \quad 2.$$
$$x+\lambda y=0. \quad 3. \qquad x+\lambda' y=0. \quad 4.$$

The double ratio of these is, taking them in the order named using the ratio,

$$\alpha = \frac{(\mu_1-\mu_3)(\mu_4-\mu_2)}{(\mu_3-\mu_2)(\mu_1-\mu_4)}, \dagger$$

where $\mu_1 = i$, $\mu_2 = -i$, $\mu_3 = \lambda$, $\mu_4 = \lambda'$, we have

$$\frac{(i-\lambda)(\lambda'+i)}{(\lambda+i)(i-\lambda')} = r+5i,$$

$$\text{or } i\lambda' - 1 - \lambda\lambda' - i\lambda = ri\lambda - r - r\lambda\lambda' - ri\lambda' - s\lambda - si - s\lambda\lambda' i + s\lambda.$$

Or, equating the real parts, and the imaginary parts, we have,

$$\lambda' - \lambda = r(\lambda - \lambda') - s(1+\lambda\lambda'),\ (\lambda-\lambda')(r+1) = s(1+\lambda\lambda').$$

$$1+\lambda\lambda' = r(1+\lambda\lambda') + s(\lambda-\lambda'),\ s(\lambda-\lambda') = (1-r)(1+\lambda\lambda').$$

$$\therefore\ \frac{\lambda-\lambda'}{1+\lambda\lambda'} = \frac{s}{1+r} = \frac{1-r}{s}.$$

And we see that r and s are restricted to the relation : $r^2+s^2=1$. If ϕ denote the angle between the lines 3 and 4,

*Compare Salmon's Conic Sections, page 133, Ex. 2.

†Clebsch, Vorlesungen ueber Geometrie, S. 38.

$$\tan\phi = \frac{s}{1+r} = \pm\sqrt{\frac{1-r}{1+r}}. \quad \therefore\ r=\cos 2\phi,\ s=\pm\sin 2\phi.$$

If we denote our double ratio of the four lines by (DR), and choose the lower sign for s, we have

$$(DR)=\cos 2\phi - i\sin 2\phi, \text{ or}$$
$$(DR)=e^{-2\phi i},$$
$$\phi=\tfrac{1}{2}i\log(DR). \quad (26)$$

We could have obtained this result in another way. If the equations of 3 and 4 were written more generally

$$ux+vy+1=0.$$
$$u'x+v'y+1=0.$$

$\tan\phi = \dfrac{uv'-u'v}{uu'+vv'}$. If $z=x+yi$, we have,

$$\log(x+yi)=\tfrac{1}{2}\log(x^2+y^2)+i\tan^{-1}(y/x).$$
$$\log(x-yi)=\tfrac{1}{2}\log(x^2+y^2)-i\tan^{-1}(y/x).$$
$$\log\frac{x+yi}{x-yi}=2i\tan^{-1}(y/x)=2i\omega, \text{ where } \tan\omega=y/x.$$

$$\omega=\tfrac{1}{2}i\log\frac{x-yi}{x+yi}.$$

Using this as a formula for expressing ϕ, we get

$$\phi=\tfrac{1}{2}i\log\left(\frac{uu'+vv'+i(u'v-uv')}{uu'+vv'-i(u'v-uv')}\right)$$

$$=\tfrac{1}{2}i\log\left(\frac{uu'+vv'+\sqrt{(uu'+vv')^2-(u^2+v^2)(u'^2+v'^2)}}{uu'+vv'-\sqrt{(uu'+vv')^2-(u^2+v^2)(u'^2+v'^2)}}\right).* \qquad (27)$$

But the expression under the logarithm is the quotient of the roots of the equation in λ,

$$u^2+v^2+2\lambda(uu'+vv')+\lambda^2(u'^2+v'^2)=0,†$$

an equation obtained by substituting in the line equation of the circular points, (11), the values $u+\lambda u'$, $v+\lambda v'$, so that the ratio of the two roots of λ is again

*First given by Laguerre: Nouvelles Annales de Math. 1853.

†See Clebsch, Vorlesungen ueber Geometrie, S. 148.

the double ratio of the four lines. Klein in the before mentioned lecture on Non-Euclidean Geometry obtains the same result in still another way. The angle between two lines is thus also defined projectively with reference to the two fixed circular points at infinity, for the double ratio of four lines is an absolute invariant under linear transformation. Some special results in angle determination may interest us.*

1. The angle that a line to either of the two circular points makes with any other line of the finite region is to be regarded as infinite. For the tangent of that angle is given by

$$\frac{\tan\psi-i}{1+i\tan\psi}=-i,$$

i, and ψ, being the tangents of the two lines. Now we have,

$$\tan x=\frac{1}{i}\tanh xi=\frac{1}{i}\,\frac{e^{2xi}-1}{e^{2xi}+1}.\quad \lim_{x=\infty}\tan x=\frac{1}{i},\ (\tan x)_{x=\infty}=-i.$$

The above angle between the two lines must therefore be regarded as an infinite one. Similarly for the line to the other circular point. By our new definition of angle, the matter is simpler still, for then in this case $\lambda=i$, or $-i$, and $(DR)=r+si=0$, or ∞, whence $\phi=\infty$.

2. Two lines are perpendicular to each other when the double ratio (26) is equal to -1, that is when the four lines form a harmonic quadrupel. For using $-\pi i$ as $\log(-1)$ we get from (26), $\phi=\frac{1}{2}\pi$. Also above, put $r=-1$, $s=0$, and obtain the same result.

3. Two lines are parallel when $r=1$, $s=0$, that is when the double ratio of the four lines is unity.

4. Two lines make an angle of 45°, or 135° when $r=0$, $s=\pm l$; that is when the double ratio is equal to $\pm i$.

5. All angles inscribed in a circle and intercepting the same arc are equal for the double ratio of four rays from some variable point in a circle to four fixed points in constant. Here the four fixed points are the two finitepoints at the ends of the arc, and the two fixed circular points at infinity. But if the double ratio is constant r and s are constant, therefore,

$$\frac{\lambda-\lambda'}{1+\lambda\lambda'}=\frac{s}{1+r}=\frac{1-r}{s}$$

is constant, and the inscribed angle is constant.

IV. Relation of the Circular Points to Non-Euclidean Geometry.

What we have established in the preceding seems to suggest the way for investigations and generalizations of the greatest importance. And such was the course of history on the analytic side of the passage from Euclidean to Non-Euc-

†Clebsch, Vorlesungen ueber Geometrie, S. 147-149.

lidean geometry. It only remained to make the generalization that, $\sum xx = \sum a_{ik} x_i x_k = 0$, being the equation of the fundamental form in point or in line coördinates as might be needed, the expression

$$\varkappa \log\left(\frac{\sum xx' + \sqrt{(\sum xx')^2 - \sum xx' . \sum x'x}}{\sum xx' - \sqrt{\sum xx' - \sum xx' . \sum x'x}} \right)$$

should be in general the distance between the points, or the angle between two lines. If $\varkappa = \frac{1}{2} i$, and $\sum xx = u^2 + v^2$ we have the ordinary Euclidean angle between two lines. If $\sum xx'$ is not equal to $u^2 + v^2$, we evidently have something quite different from that angle, $\varkappa$ times the logarithm of the double ratio of the two lines and the pair of tangents to the conic from their point of intersection.

The derivation of the Euclidean distance formula is not so simple, a case of limits being involved. According as this fundamental conic is an actual one, a point pair, or an imaginary one, we get hyperbolic, parabolic, or elliptic metrical determination. Cayley seems to have given the first valuable suggestions tending towards analytic methods. Klein has built up an admirable analytic treatment, using what he calls the "Cayley'schen Maassbestimmung" as a basis. In his illustrations of elliptic, and hyperbolic geometry of the plane, he uses as fundamental conics $x^2 + y^2 = -r^2$, and $x^2 + y^2 = r^2$ respectively. It is interesting to note that the square of the element of length in each is,

$$ds_1^2 = \frac{dx^2 + dy^2 + \frac{(ydx - xdy)^2}{r^2}}{\left(1 + \frac{x^2 + y^2}{r^2}\right)^2}, \qquad ds_3^2 = \frac{dx^2 + dy^2 - \frac{(ydx - xdy)^2}{r^2}}{\left(1 - \frac{x^2 + y^2}{r^2}\right)^2}.$$

Now if r becomes indefinitely great, we have as the limit of both ds_1^2 and ds_3^2, $ds_2^2 = dx^2 + dy^2$, the square of the element of length in the ordinary Euclidean plane. This affords incidentally confirmation of our proposition under II. C, 2. that when the radius of a circle whose center is in the finite region becomes indefinitely great, the circle degenerates into the two circular points at infinity.